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SHORTER COMMUNICATIONS

ON DIFFERENTIAL METHODS FOR RADIANT HEAT TRANSFER

S. C. TRAUGOTT and K. C. WANG
Research and Development Department, The Martin Company, Baltimore, Maryland

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In a recent paper Adrianov and Polyak [1] point out that there exist differential approximations to the exact integral expression for radiative heat flux, and that these can give excellent results. We heartily concur with the authors and wish to point out that one of the approximations discussed, the Schuster–Schwarzschild method, can easily be modified to be even better than indicated in their discussion.

We use the nomenclature of [1] throughout. Equations (3) of reference [1] for a grey gas in local thermodynamic equilibrium are:

$$\left.\begin{aligned} \frac{dE_+}{dx} &= 2kE_0 - km_+E_+ \\ \frac{dE_-}{dx} &= -2kE_0 + km_-E_- \end{aligned}\right\} \qquad (1)$$

$$m_+ = \frac{\int_{2\pi} J_+ d\omega}{\int_{2\pi} J_+ \cos\theta \, d\omega}$$

$$m_- = \frac{\int_{-2\pi} J_- d\omega}{\int_{-2\pi} J_- \cos\theta \, d\omega}$$

Equations (1) follow from an integration over angle of the radiative transfer equation, with m_+ and m_- functions of x. The Schuster–Schwarzschild method of reference [1] results from putting $m_+ = m_- = 2$. Let us assume only that $m_+ = m_- = \bar{m}$ without yet specifying a value for $\bar{m}$.

Differentiating equations (1), with $d\tau = k\,dx$, leads to

$$\frac{d^2}{d\tau^2}(E_+ - E_-) = 4\frac{dE_0}{d\tau} + \bar{m}^2(E_+ - E_-).$$

Since $q = E_+ - E_-$ this becomes

$$\frac{d^2q}{d\tau^2} = 4\frac{dE_0}{d\tau} + \bar{m}^2 q. \qquad (2)$$

Equation (2), not considered in reference [1], is a single second order equation for q, the net radiative flux, whereas equations (1) form a system of two first order equations for the two components of this flux along x and $-x$. Which form is more useful will depend on whether the boundary conditions are on the separate flux components or on the net flux. Equation (2) shows immediately that for small optical thickness the transparent limit $dq/d\tau = 4E_0$ can be recovered by dropping the second term on the right, the absorption term. For large optical thickness the Rosseland diffusion limit can be obtained by equating the first and second terms on the right and by choosing $\bar{m}^2 = 3$. Choosing $\bar{m}^2 = 4$ as in [1] leads to an error in the large τ limit, with q too small by a factor 3/4.

Equation (2) with $\bar{m}^2 = 3$ can be obtained in a number of ways and is equivalent to the Milne–Eddington approximation generalized to radiative-non-equilibrium (see Goody [2] and Viskanta and Grosh [3]). It may be of interest to point out that a similar equation has been given by Rumynskii [4] which unfortunately also fails in the opaque limit.

For radiative equilibrium it is pointed out by Sobolev [5] that a discrepancy of 3/4 exists in the large τ limit for the Schuster–Schwarzschild method compared to the Milne–Eddington approximation, which becomes exact. But this factor is precisely the error of the Schuster–Schwarzschild calculation of Adrianov and Polyak [1] for large optical thickness when compared to an exact numerical calculation, see their Figs. 1 and 2. We therefore suggest that with $\bar{m} = \sqrt{3}$ in equations (1) the results of reference [1] with this very simple model would have been even better, attaining the proper limit for large τ.

With this modification their Schuster–Schwarzschild method, equation (4) of [1], becomes identical to their gradient method, equation (17) of [1]. It is of interest to note that Adrianov and Polyak's equation (17), for the particular example considered, is precisely that recently given by Probstein [6], see his equation (11). Probstein obtains his result from the Rosseland diffusion approximation together with a temperature jump boundary condition, essentially modifying the opaque limit to be correct also in the transparent limit.

As may be seen either from Fig. 1 of [1] or Fig. 1 of [6], excellent agreement with numerical calculations based on the exact integral expressions can be obtained.

In summary, it appears that for those problems in which the nature of the boundary conditions makes a choice of the separate heat flux components as variables logical, good results should be obtainable from equations (1) with $m_+ = m_- = \sqrt{3}$.

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AN ENGINEERING APPROXIMATION FOR RESISTANCES OF CERTAIN TWO-DIMENSIONAL CONDUCTORS

ELDON L. KNUTH*

Department of Engineering, University of California, Los Angeles

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MODEL DESCRIPTION

The model studied was a two-dimensional conductor of thickness l containing a cut of depth m and width n (cf. sketch in Fig. 1). The conductivity was taken to be constant. The problem, from the viewpoint of the designer, may be stated: Predict the length L' of an equivalent conductor of uniform thickness l having the same gross conduction properties as a conductor of length L containing a cut such as depicted in Fig. 1. In order to obtain the solution to this problem, one may consider either of the following cases:

1. Insulated surfaces, flow along the conductor.
2. Constant-potential surfaces, flow across the conductor.

The constant-potential and flow lines of the first case are respectively the flow and constant-potential lines of the second case. The cut increases the length of the equivalent conductor, in the first case, by increasing the resistance to flow and, in the second case, by decreasing the resistance to flow. For convenience, the discussion during the bulk of the paper is in terms of the first case.

ANALYTICAL STUDIES

Analyses of potential flows in systems with polygonal boundaries are facilitated by use of the Schwarz–Christoffel transformation [1, p. 445]. Using this method, Schofield [2] finds for the conductor being considered

$$\frac{L'}{l} = \frac{L-n}{l} + \frac{n}{l-m} + \frac{X}{l} \tag{1}$$

$$\frac{X}{l} = \frac{2}{\pi}\ln\left[\frac{2\,\mathrm{sn}\alpha}{\mathrm{cn}\alpha\;\mathrm{dn}\alpha\;\mathrm{sn}2\alpha}\,\frac{\Theta(0)}{\Theta(2\alpha)}\right] + \frac{n}{l}\left(1-\frac{\alpha}{K}\right) - \frac{n}{l-m} \tag{2}$$

where the constants α and K are to be determined by solving simultaneously

$$\frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha}{\mathrm{dn}\alpha} - Z(\alpha) = \frac{\pi}{4K}\frac{n}{l} \tag{3}$$

$$\frac{k^2\,\mathrm{sn}\alpha\,\mathrm{cn}\alpha}{\mathrm{dn}\alpha} - Z(\alpha) = \frac{\pi}{2K'}\left(\frac{\alpha}{K} - \frac{m}{l}\right) \tag{4}$$

(see, e.g. [3] for a discussion of elliptic integrals which is brief and yet is adequate for the present discussion). The first two terms appearing in the right-hand side of equation (1), i.e. $(L-n)/l$ and $n/(l-m)$, represent respectively the resistances of the parts with thicknesses l and $l-m$ to one-dimensional flows; the term X/l represents the additional resistance due to the fact that the flow paths are curved in the vicinity of the junctions of the parts with differing thicknesses. The probability that a design engineer would take the time required to solve simultaneously equations (3) and (4) appears to be small. Hence, a convenient approximate solution to equations (2–4) is sought.

A useful guide in the search for this approximate solution is provided by the physical interpretation of the term X/l—"the additional resistance due to the fact that the flow paths are curved". Curvature of the flow paths is due primarily to the fact that m/l is non-zero. Hence one is motivated to examine the term X/l for the limiting

* Associate Professor and Head of Chemical, Nuclear and Thermal Division.